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Boson realisation of the Lie algebra F_4 and non-trivial zeros of $6j$ symbols

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Abstract. By decomposing the adjoint and lowest dimensional representations of the exceptional Lie algebra F_4 in the reduction $F_4 \to SO_3$, a boson realisation of a F_4 generator basis is established on account of a standard tensor operator formalism. Such a basis clearly exhibits the non-trivial vanishing of two $6j$ coefficients (discarding Regge symmetries), a property which is closely related to the possible embedding of F_4 into SO_{26}. Prospects for the occurrence of more non-trivial zeros, related to exceptional groups of higher rank, are indicated.

1. Introduction

The existence of a class of zeros of the Racah $6j$ coefficients, which may be called non-trivial or structural zeros since they do not result from violation of the triangle conditions, has been discussed recently by Biedenharn and Louck (1981b). In their book, an extensive table is included listing over 1400 such zeros of the $6j$ symbol. More striking, however, is the fact that until now, if one disregards Regge symmetries, a significant explanation has been presented for only two of these accidental zeros. The first example is that of the coefficient

$$\begin{Bmatrix} 2 & 2 & 2 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{Bmatrix},$$

the vanishing of which is rather easily explained within the quasi-spin model. The second and better known example is that of the $6j$ symbol

$$\begin{Bmatrix} 5 & 5 & 3 \\ 3 & 3 & 3 \end{Bmatrix}$$

of which the zero value elucidates the embedding of the exceptional Lie algebra G_2 in the algebra SO_7. This fact can be readily verified if one realises the generators of SO_7 as tensor operators with respect to the SO_3 subgroup in the chain $SO_7 \supset G_2 \supset SO_3$. Biedenharn and Louck (1981b) remark that it would be of considerable interest to discuss the remaining exceptional groups by similar explicit results. It is precisely the aim of the present paper to make a first decisive step in this direction.

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The next smallest exceptional group after G_2 is F_4, and some partial results have been obtained already for this group. More precisely a (boson) realisation of the F_4 generators has been constructed by Wadzinski (1969) by expressing them in terms of SO_3 tensor operators. This construction, however, is not the simplest one since it was, moreover, required by Wadzinski that the generator basis should make apparent the SO_9 subgroup structure contained in F_4. Such a condition implied the introduction of four types of bosons, each carrying a different angular momentum, whereas the tensor operator components needed to express the F_4 generators form a subset of a U_{26} infinitesimal operator basis. Hence, we can summarise that Wadzinski's treatment of F_4 is related to the $U_{26} \supset F_4 \supset SO_9 \supset SO_3$ chain of groups. Disappointing, however, is the fact that within this basis of the F_4 algebra, it is impossible to explain any non-trivial zero of the $6j$ symbol. In turn, as was shown by Wadzinski (1969), many relations between $6j$ coefficients follow from it. Caused partially by the fact that the number of bosons is too large, relations are found instead of structural zeros; we believe it is possible to explain some of the latter in realising the F_4 algebra by means of two bosons only. This can be accomplished by considering the reduction of F_4 straight into SO_3, a point of view which will turn out to be related to the chain $SO_{26} \supset F_4 \supset SO_3$, that is, to the embedding of F_4 in SO_{26}. Indeed, explicit calculations will show that two new non-trivial zeros of the $6j$ coefficients can be explained, whereas the use of Regge symmetries enlarges this number up to eight contained in the table of Biedenharn and Louck (1981b).

Let us mention finally that in labelling irreducible representations we shall follow the same conventions as in the book by McKay and Patera (1981), with the exception of SO_3 representations, which we shall label by half the number that they use.

2. Tensor operators

SO_3 tensor operators are defined by means of reduced matrix elements (Judd 1963, Wadzinski 1969):

$$\langle \tau_2' l_2' \| v^k(\tau_2 l_2, \tau_1 l_1) \| \tau_1' l_1' \rangle = [k]^{1/2} \delta_{\tau_2' \tau_2} \delta_{l_2' l_2} \delta_{\tau_1' \tau_1} \delta_{l_1' l_1} \tag{2.1}$$

where l and k are SO_3 representation labels, $[k] = 2k+1$ is the dimension of the tensor representation and τ is an additional label to distinguish states with the same l. These operators obey the following commutation relations:

$$\begin{aligned}
&[v^{k_1}_{q_1}(\tau_1 l_1, \tau_2 l_2), v^{k_2}_{q_2}(\tau_3 l_3, \tau_4 l_4)] \\
&\quad = \sum_{k_3, q_3} \{[k_1][k_2][k_3]\}^{1/2} \begin{pmatrix} k_1 & k_2 & k_3 \\ q_1 & q_2 & -q_3 \end{pmatrix} (-1)^{2l_4 + l_3 - l_2 - q_3} \\
&\quad \times \left(\delta_{\tau_2 \tau_3} \delta_{l_2 l_3} (-1)^{k_1+k_2+k_3+l_1+l_2+l_3+l_4} \begin{Bmatrix} k_1 & k_2 & k_3 \\ l_4 & l_1 & l_3 \end{Bmatrix} v^{k_3}_{q_3}(\tau_1 l_1, \tau_4 l_4) \right. \\
&\quad \left. - \delta_{\tau_1 \tau_4} \delta_{l_1 l_4} \begin{Bmatrix} k_1 & k_2 & k_3 \\ l_3 & l_2 & l_1 \end{Bmatrix} v^{k_3}_{q_3}(\tau_3 l_3, \tau_2 l_2) \right).
\end{aligned} \tag{2.2}$$

Note that in principle, l can take integer or half-integer values. Realisations in terms of tensor operators with integer l will be called boson realisations hereafter. Such a realisation is established for any compact semi-simple Lie algebra H by writing the generators of H as

$$G_{\gamma kq} = \sum_{\beta_2 l_2 \beta_1 l_1} g[\gamma k; \beta_1 l_1, \beta_2 l_2] v_q^k(\beta_1 l_1, \beta_2 l_2) \tag{2.3}$$

whereby (γk) are the SO_3 representations in which the adjoint representation of H decomposes, γ being used to distinguish between similar representations. A method to calculate the coefficients $g[\ldots]$ in (2.3) has been discussed by Wadzinski (1969). As an application this author constructed a realisation of the F_4 generators which has the peculiarity that it makes the SO_9 subalgebra contained in F_4 apparent. Since we want to further compare his approach with ours, we give Wadzinski's results here (with corrections). The 52-dimensional adjoint representation (1 0 0 0) of F_4 decomposes into SO_9 representations (0 1 0 0) and (0 0 0 1). In turn these reduce into the SO_3 representations (7), (5), (3), (1) and (5), (2) respectively. The non-trivial lowest dimensional F_4 representation (0 0 0 1) decomposes into the SO_9 representations (0 0 0 1), (1 0 0 0), (0 0 0 0) which respectively decompose into the SO_3 representations (5), (2) and (4), (0). As a consequence four bosons are introduced which in spectroscopic notation are written as h, d, g and s. The F_4 generator realisation is given by

$$\begin{aligned}
G_{1q} &= 3^{-1/2} v_q^1(\mathrm{gg}) + \tfrac{1}{3} 2^{-1/2} v_q^1(\mathrm{dd}) + \tfrac{1}{3}(11/2)^{1/2} v_q^1(\mathrm{hh}) \\
G_{3q} &= 3^{-1/2} v_q^3(\mathrm{gg}) + \tfrac{1}{4}(11/3)^{1/2} v_q^3(\mathrm{dd}) - \tfrac{1}{12} 13^{1/2} v_q^3(\mathrm{hh}) + (-1)^{\beta}(5/12)(v_q^3(\mathrm{dh}) - v_q^3(\mathrm{hd})) \\
G_{a5q} &= 3^{-1/2} v_q^5(\mathrm{gg}) + \tfrac{1}{3} v_q^5(\mathrm{hh}) - (-1)^{\beta} \tfrac{1}{3}(5/2)^{1/2}(v_q^5(\mathrm{dh}) - v_q^5(\mathrm{hd})) \\
G_{7q} &= 3^{-1/2} v_q^7(\mathrm{gg}) - \tfrac{1}{6}(17)^{1/2} v_q^7(\mathrm{hh}) - (-1)^{\beta} \tfrac{1}{6}(7/2)^{1/2}(v_q^7(\mathrm{dh}) - v_q^7(\mathrm{hd})) \\
G_{2q} &= \tfrac{1}{2} 2^{-1/2}[v_q^2(\mathrm{sd}) + (-1)^{\delta} v_q^2(\mathrm{ds})] + (-1)^{\varepsilon} \tfrac{1}{4}(7/3)^{1/2}[v_q^2(\mathrm{gd}) + (-1)^{\delta} v_q^2(\mathrm{dg})] \\
&\quad - (-1)^{\varepsilon+\beta} \tfrac{1}{4}(11/3)^{1/2}[v_q^2(\mathrm{gh}) - (-1)^{\delta} v_q^2(\mathrm{hg})] \\
G_{b5q} &= \tfrac{1}{2} 2^{-1/2}[v_q^5(\mathrm{sh}) + (-1)^{\delta} v_q^5(\mathrm{hs})] + (-1)^{\varepsilon+\beta} \tfrac{1}{4}(5/3)^{1/2}[v_q^5(\mathrm{gd}) - (-1)^{\delta} v_q^5(\mathrm{dg})] \\
&\quad + (-1)^{\varepsilon} \tfrac{1}{4}(13/3)^{1/2}[v_q^5(\mathrm{gh}) + (-1)^{\delta} v_q^5(\mathrm{hg})].
\end{aligned} \tag{2.4}$$

Hereby β, δ and ε can be attributed arbitrary but fixed integer values. Also, the first four lines of (2.4) constitute a realisation of the SO_9 subalgebra. Furthermore, the set of operators $v_q^k(st)$, where s and t stand for g, h, d or s and with k and q running through all acceptable values, constitutes a basis of the U_{26} Lie algebra. Hence, the realisation (2.4) makes the embedding of F_4 into U_{26} explicit.

3. Two-boson realisation of F_4

F_4 possesses a maximal SO_3 subalgebra (Dynkin 1957a, b). The corresponding branching rules for F_4 representations are found in the tables of McKay and Patera (1981). So we immediately learn that (1000), the adjoint representation of F_4, decomposes into the representations (11), (7), (5) and (1) of that maximal SO_3 subalgebra, whereas (0001), the 26-dimensional representation of F_4, reduces into (8) and (4). Hence, instead of the four bosons in the preceding section, we need only two for which we reserve in the spectroscopic notation the characters l and g respectively. Proceeding as in § 2 we succeeded in casting the F_4 algebra into the following form:

$$
\begin{aligned}
G_{1q} &= 2\sqrt{3}\sqrt{5}v_q^1(\mathrm{gg}) + 2\sqrt{2}\sqrt{3}\sqrt{17}v_q^1(\mathrm{ll}) \\
G_{5q} &= \frac{6\sqrt{3}\sqrt{17}}{\sqrt{13}}v_q^5(\mathrm{gg}) + \frac{6\sqrt{2}\sqrt{19}}{\sqrt{13}}v_q^5(\mathrm{ll}) + (-1)^\alpha \frac{12\sqrt{2}\sqrt{5}}{\sqrt{13}}(v_q^5(\mathrm{gl}) + v_q^5(\mathrm{lg})) \\
G_{7q} &= \frac{\sqrt{3}\sqrt{11}\sqrt{19}}{\sqrt{13}}v_q^7(\mathrm{gg}) - \frac{\sqrt{3}\sqrt{17}\sqrt{23}}{\sqrt{13}}v_q^7(\mathrm{ll}) \\
&\quad + (-1)^\alpha \frac{3\sqrt{2}\sqrt{7}\sqrt{17}}{\sqrt{13}}(v_q^7(\mathrm{gl}) + v_q^7(\mathrm{lg})) \\
G_{11q} &= 3\sqrt{2}\sqrt{3}\sqrt{5}v_q^{11}(\mathrm{ll}) - (-1)^\alpha 3\sqrt{11}(v_q^{11}(\mathrm{gl}) + v_q^{11}(\mathrm{lg})).
\end{aligned}
\tag{3.1}
$$

Here too, the integer α may be freely chosen. Furthermore, we notice that in (3.1) a tensor operator with boson content of mixed type is always part of a sum of which the other summand is a tensor operator differing from the former by an interchange of the two bosons. Also, in (3.1) only tensor operators of odd rank occur. These two properties make the embedding of F_4, as given by (3.1), into SO_{26} explicit (Elliott 1958, Judd 1963).

As a supplementary verification of the validity of (3.1) we have transformed (3.1) into the Cartan–Weyl standard form after which we verified that the obtained root structure is in agreement with the well known result given in many textbooks.

4. Non-trivial zeros

An interesting property of the realisation (3.1) is that in the progression of generators some odd-rank tensors are missing which can exist on grounds of angular momentum coupling and which occur in SO_{26}. We shall later demonstrate that this peculiarity, which reflects the possible embedding of F_4 into SO_{26} and which is very similar to what appears in the $G_2 \subset SO_7$ case, is one of the origins for the occurrence of non-trivial zeros of $6j$ coefficients. But let us first point out that the missing of even-rank tensors in (3.1) is not at all related to structural zeros and does not even give rise to relations between $6j$ symbols. Indeed, within the algebra (3.1) it is trivially excluded that even-rank tensors could be generated by the commutation of generators. As an example take the operator $v_q^k(\mathrm{gg})$ with k even and $0 \leqslant k \leqslant 8$. As may be verified from equation (2.2), such an operator could only be obtained by working out the commutators $[v_{q_1}^{k_1}(\mathrm{gg}), v_{q_2}^{k_2}(\mathrm{gg})]$, $[v_{q_1}^{k_1}(\mathrm{gl}), v_{q_2}^{k_2}(\mathrm{lg})]$ and $[v_{q_1}^{k_1}(\mathrm{lg}), v_{q_2}^{k_2}(\mathrm{gl})]$ where k_1, k_2 equal 1, 5, 7 or 11 and $q_1 + q_2 = q$. But, $k_1 + k_2 + k$ being even, it is obvious from (2.2) that the first commutator type cannot produce a term $v_q^k(\mathrm{gg})$, whereas such a term arising from the second-type commutator is exactly cancelled by the similar term arising from the third-type commutator. This follows from the property that in the generator basis (3.1) the operators $v_q^k(\mathrm{gl})$ and $v_q^k(\mathrm{lg})$ always appear with the same coefficient. The same reasoning applies to operators $v_q^k(\mathrm{ll})$ with k even and $0 \leqslant k \leqslant 16$ and to operator combinations $v_q^k(\mathrm{gl}) + v_q^k(\mathrm{lg})$ with k even and $4 \leqslant k \leqslant 12$, all of which could be found as constituents of an operator G_{kq} with k even.

Let us now return to the fact that G_{3q} is missing in the F_4 algebra (3.1). A possible constituent of G_{3q} is $v_q^3(\mathrm{gg})$, for which we investigate how it could be generated from commutators of F_4 generators. Again the only types of commutators which can produce $v_q^3(\mathrm{gg})$ are the three mentioned above. Since now k_1+k_2+3 is odd, the first type indeed generates a $v_q^3(\mathrm{gg})$ term as can be verified from (2.2). The second and third types also both generate that operator with the same coefficient and hence the two contributions add instead of cancelling each other as before. As a consequence, for each valid set of k_1 and k_2 values a relation between two differept $6j$ symbols in general follows. There is, however, one exception, namely when $k_1=k_2=11$. Indeed, since $v_q^{11}(\mathrm{gg})$ does not appear in G_{11q}, because it does not even exist on account of coupling restrictions, only the equal contributions coming from the second- and third-type commutators survive. These contributions are proportional to the $6j$ coefficient

$$\begin{Bmatrix} 11 & 11 & 3 \\ 4 & 4 & 8 \end{Bmatrix}$$

which necessarily has to vanish. The coefficient is not zero by triangle condition violation and is, therefore, our first new example of a group-theoretical explanation of a structural zero. However, $v_q^3(\mathrm{gg})$ is the only constituent of G_{3q} which leads to such an explanation. For instance, in order to investigate $v_q^3(\mathrm{ll})$ it suffices to replace in the entire reasoning g by l and vice versa. In general, relations between two $6j$ symbols follow. The only exception is now $k_1=k_2=1$ wherefore $v_q^1(\mathrm{gl})$ and $v_q^1(\mathrm{lg})$ are non-existent, but the related $6j$ symbol is trivially zero.

The next case to be considered is that of G_{9q}. We shall not repeat here an analogous argumentation, but leave it to the reader to verify that again one structural zero can be explained, namely

$$\begin{Bmatrix} 11 & 11 & 9 \\ 8 & 4 & 8 \end{Bmatrix}=0.$$

The entries of the symbol indicate that the explanation should be sought in the impossibility to generate an operator $v_q^9(\mathrm{gl})$ or $v_q^9(\mathrm{lg})$ in the commutator $[G_{11q}, G_{11q'}]$. Finally, the missing of G_{13q} and G_{15q} in (3.1) is not associated with any new structural zero.

Summarising, it has been proven that the realisation (3.1) provides a basis for the explanation of two non-trivial zeros of the $6j$ coefficient. In fact, it should be noted that even more of those listed in the tables of Biedenharn and Louck (1981b) can be explained. Indeed, on account of Regge symmetries (see, e.g., Biedenharn and Louck 1981a) we immediately obtain

$$\begin{Bmatrix} 11 & 11 & 3 \\ 4 & 4 & 8 \end{Bmatrix}=\begin{Bmatrix} 11 & 10 & 2 \\ 4 & 5 & 9 \end{Bmatrix}=0 \tag{4.1}$$

$$\begin{aligned}\begin{Bmatrix} 11 & 11 & 9 \\ 8 & 4 & 8 \end{Bmatrix}&=\begin{Bmatrix} 12 & 11 & 8 \\ 5 & 8 & 7 \end{Bmatrix}=\begin{Bmatrix} 11 & 10 & 10 \\ 4 & 9 & 7 \end{Bmatrix}=\begin{Bmatrix} 13 & 9 & 9 \\ 6 & 8 & 6 \end{Bmatrix}\\ &=\begin{Bmatrix} 13 & 10 & 8 \\ 6 & 7 & 7 \end{Bmatrix}=\begin{Bmatrix} 12 & 10 & 9 \\ 5 & 9 & 6 \end{Bmatrix}=0.\end{aligned} \tag{4.2}$$

For the sake of completeness, let us note that attention is rarely drawn to the fact that the well known structural zero related to $G_2 \subset SO_7$ also entails another one, namely

$$\begin{Bmatrix} 5 & 5 & 3 \\ 3 & 3 & 3 \end{Bmatrix}=\begin{Bmatrix} 5 & 4 & 4 \\ 3 & 4 & 2 \end{Bmatrix}=0. \tag{4.3}$$

As a conclusion we can say that we have succeeded in explaining eight new non-trivial zeros in the same way as the two mentioned in (4.3) which have been known for a long time.

5. Discussion and outlook

The foregoing analysis shows that we can expect to explain more non-trivial zeros if two conditions can be satisfied in the construction of a SO_3 tensor operator basis for the generators of a classical Lie algebra. The first is that some tensor operators should be missing exactly in the way we missed G_q^3 and G_q^9 in the two-boson realisation here. A quick search through the tables of McKay and Patera (1981) shows that this is most likely to happen for the exceptional algebras. A second condition is that at least one tensor operator which is absent in the generator algebra appears only in the right-hand side of one commutator of operators which arise in the algebra as generator constituents (more commutators can be accepted if symmetry arguments apply). It is clear that in order to achieve that aim, the number of different bosons (or, more generally, different SO_3 representation labels) should be kept minimal and certainly degeneracies of l multiplicity should be avoided. Hence, we want not only to reduce into SO_3 representations the first acceptable lowest dimensional representation of the algebra under consideration but, moreover, we prefer to select a maximal SO_3 subalgebra if it exists, and if choice remains we give preference to the principal SO_3 subalgebra (Dynkin 1957a, b). Otherwise, we must indicate a chain ending with an SO_3 such that the second condition is maximally satisfied.

Both conditions at first sight seem rather hard to satisfy. Nevertheless, we can predict that in the chains $E_6 \supset F_4 \supset SO_3$ and $E_7 \supset SO_3$ new zeros can be explained.

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